Math 247A Lecture 20 Notes

Daniel Raban

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1 Littlewood-Paley Projections and Khinchine's Inequality

1.1 Bernstein properties of Littlewood-Paley projections

Last time, we were proving properties of Littlewood-Paley projections.

Theorem 1.1.

1. $\|f_n\|_p + \|f_{\leq N}\|_p \lesssim \|f\|_p$ *uniformly in* N *and for* $1 \leq p \leq \infty$.
2. $|f_N(x)| + |f_{\leq N}(x)| \lesssim Mf(x)$.
3. *For* $f \in L^p$ *with* $1 < p < \infty$, *we have* $f \overset{L^p}{=} \sum_{N \in 2^{\mathbb{Z}}} f_N$.
4. *(Bernstein's inequality) For* $1 \leq p \leq q \leq \infty$,
$$\|f_N\|_q \lesssim N^{d/p-d/q} \|f_N\|_p$$
$$\|f_{\leq N}\|_q \lesssim N^{d/p-d/q} \|f_{\leq N}\|_p.$$
5. *(Bernstein) For* $1 \leq p \leq \infty$ *and* $s \in \mathbb{R}$,
$$\||\nabla|^s f_N\|_p \sim N^s \|f_N\|_p.$$
In particular, for $s > 0$ *and* $1 \leq p \leq \infty$,
$$\||\nabla|^s f_{\leq N}\|_p \lesssim N^s \|f_{\leq N}\|_p.$$
$$\|f_{>N}\|_p \lesssim N^{-s} \||\nabla|^s f_{>N}\|_p.$$

We proved properties (1) to (3) last time.

Proof. Here is 4: We have $f_N = f * N^d \psi^\vee(N\,\cdot)$, so by Young's inequality,

$$\begin{aligned}\|f_n\|_q &\lesssim \|f\|_p \cdot \|N^d \psi^\vee(N\,\cdot)\|_{qp/(qp+p-q)} \\ &\lesssim \|f\| N^{d-d(1+1/q-1/p)} \\ &\lesssim N^{d/p-d/q}\|f\|_p\end{aligned}$$

To recover f_N on the RHS, we use a common trick. Let $\widetilde{\psi}(\xi) = \psi(2\xi) + \psi(\xi) + \psi(\xi/2)$, $\widetilde{\psi}_N(\xi) = \widetilde{\psi}(\xi/N)$, and define the fattened LIttlewood-Paley projection

$$\widehat{\widetilde{P}_N f}(\xi) = \widehat{f}(\xi) \cdot \widetilde{\psi}_N(\xi).$$

Note that $\widetilde{P}_N P_n = P_n$ since $\widetilde{\psi} \equiv 1$ on $\operatorname{supp}\psi$. Write

$$f_N = \widetilde{P}_N f = f_n * [N^d \widetilde{\psi}^\vee(N\,\cdot)]$$

and argue as before. The same argument gives $\|f_{\leq N}\|_q \leq N^{d/p-d/q}\|f_{\leq N}\|_p$. (We use $P_{\leq 4N} P_{\leq N} = P_{\leq N}$.)

Here is 5: Note that

$$\begin{aligned}|\nabla|^s f_N &= [(2\pi|\xi|)^s \psi_N(\xi)]^\vee * f \\ &= N^s \left[\left(\frac{2\pi|\xi|}{N}\right)^s \psi(\xi/N))\right]^\vee * f.\end{aligned}$$

Let

$$\chi(\xi) = (2\pi|\xi|)^s \psi(\xi) \in C_c^\infty(\mathbb{R}^d \setminus \{0\}), \qquad \chi_N(\xi) = \chi(\xi/N).$$

Then $|\nabla|^s f_N = N^s [N^d \chi^\vee(N\,\cdot)] * f$. So

$$\begin{aligned}\||\nabla|^s f_N\|_p &\lesssim N^s \|f\|_p \underbrace{\|N^d \chi_\vee(N\,\cdot)\|}_{=\|\chi^\vee\|_1} \\ &\lesssim N^s \|f\|_p.\end{aligned}$$

Using the fattened Littlewood-Paley projection P_N, we get

$$\||\nabla|^s f_N\|_p \lesssim N^s \|f_n\|_p.$$

On the other hand,

$$\|f_n\|_p = \||\nabla|^{-s}|\nabla|^s f_n\|_p \lesssim N^{-s}\||\nabla|^s f_N\|_p.$$

Finally, for $s > 0$,

$$\||\nabla|^s f_{\leq N}\|_p \lesssim \sum_{M \leq N} \||\nabla|^s f_M\|_p$$

$$\lesssim \sum_{M \le N} M^s \underbrace{\|f_M\|_p}_{\lesssim \|f\|_p}$$
$$\lesssim N^s \|f\|_p.$$

For high frequencies,

$$\begin{aligned}
\|f_{>N}\|_p &\lesssim \sum_{M>N} \|f_M\|_p \\
&\lesssim \sum_{M>N} M^{-s} \underbrace{\||\nabla|^s f_M\|_p}_{\lesssim \||\nabla|^s f\|_p} \\
&\lesssim N^{-s} \||\nabla|^s f\|_p.
\end{aligned}$$

□

1.2 Khinchine's inequality

Lemma 1.1 (Khinchine's inequality)**.** *Let $\{X_n\}_{n\ge1}$ be independent, identically distributed random variables with $X_n = \pm 1$ with equal probability. Let $\{c_n\}_{n\ge1} \subseteq \mathbb{C}$ and $0 < p < \infty$. Then*

$$\mathbb{E}\left[\left|\sum_{n\ge1} c_n X_n\right|^p\right]^{1/p} \sim_p \sqrt{\sum_{n\ge1} |c_n|^2}.$$

One way to think about this is that a random variable's "size" is given by its variance. For $p = 2$,

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{n\ge1} c_n X_n\right|^2\right] &= \mathbb{E}[\left(\sum c_n X_n\right)\left(\sum \bar{c}_m X_m\right)] \\
&= \sum |c_n|^2\, \mathbb{E}[X_n^2] + \sum_{n\ne m} c_n \bar{c}_m \underbrace{\mathbb{E}[X_n X_m]}_{\to 0} \\
&= \sum |c_n|^2.
\end{aligned}$$

So this basically says that this orthogonality persists, even in an L^p sense.

Proof. Without loss of generality, we may assume $c_n \in \mathbb{R}$.

$$\mathbb{E}\left[\left|\sum_{n\ge1} c_n X_n\right|^p\right] = p\int_0^\infty \lambda^p \mathbb{P}\left(\left|\sum c_n X_n\right| > \lambda\right)\frac{d\lambda}{\lambda}$$

By Chebyshev,

$$\mathbb{P}\left(\sum c_n X_n > \lambda\right) \le e^{-\lambda t}\, \mathbb{E}[e^{t\sum c_n X_n}]$$

$$
\begin{aligned}
&= e^{-\lambda t}\,\mathbb{E}\left[\prod_n e^{tc_nX_n}\right] \\
&= e^{-\lambda t}\prod_n \mathbb{E}[c^{tc_nX_n}] \\
&= e^{-\lambda t}\prod_n \frac{e^{tc_n}+e^{-tc_n}}{2} \\
&= e^{-\lambda t}\prod_n \cosh(tc_n)
\end{aligned}
$$

Use that $\cosh x \le e^{x^2/2}$.

$$
\begin{aligned}
&= e^{-\lambda^t}\prod_n e^{t^2c_n^2/2} \\
&= e^{-\lambda t + t^2/2(\sum c_n^2)}.
\end{aligned}
$$

Choose t such that $\lambda t = t^2\sum c_n^2$; so $t = \lambda/\sum c_n^2$. We get

$$
\mathbb{P}\left(\sum c_nX_n > \lambda\right) \le e^{-\lambda^2/(2\sum c_n^2)}.
$$

The same argument gives

$$
\mathbb{P}\left(\sum c_nX_n < -\lambda\right) \le e^{-\lambda t}\,\mathbb{E}[e^{-t\sum c_nX_n}] \le e^{-\lambda^2/(2\sum c_n^2)}.
$$

So we have

$$
\begin{aligned}
\mathbb{E}\left[\left|\sum_{n\ge 1} c_nX_n\right|^p\right] &= p\int_0^\infty \lambda^p\,\mathbb{P}\left(\left|\sum c_nX_n\right| > \lambda\right)\frac{d\lambda}{\lambda} \\
&\lesssim_p \int_0^\infty \lambda^p e^{-\lambda^2/(2\sum c_n^2)}\frac{d\lambda}{\lambda}
\end{aligned}
$$

Make the change of variables $\beta = \lambda/\sqrt{\sum c_n^2}$.

$$
\lesssim_p \left(\sum c_n^2\right)^{p/2}\underbrace{\int_0^\infty \beta^p e^{-\beta^2/2}\frac{d\beta}{\beta}}_{\lesssim_p 1}.
$$

For the other inequality, for $1 < p < \infty$,

$$
\sum |c_n|^2 = \mathbb{E}\left[\left|\sum c_nX_n\right|^2\right]
$$

$$\lesssim \mathbb{E}\left[\left|\sum c_n X_n\right|^p\right]^{1/p} \underbrace{\mathbb{E}\left[\left|\sum c_n X_n\right|^{p'}\right]^{1/p'}}_{\lesssim\sqrt{\sum |c_n|^2}},$$

which gives us

$$\sqrt{\sum c_n^2} \lesssim \mathbb{E}\left[\left|\sum c_n X_n\right|^p\right]^{1/p}.$$

For $0 < p \leq 1$, we use Cauchy-Schwarz instead:

$$\begin{aligned}\sum |c_n|^2 &= \mathbb{E}\left[\left|\sum c_n X_n\right|^2\right] \\ &= \mathbb{E}\left[\left|\sum c_n X_n\right|^{p/2}\left|\sum c_n X_n\right|^{2-p/2}\right] \\ &\lesssim \mathbb{E}\left[\left|\sum c_n X_n\right|^p\right]^{1/2} \underbrace{\mathbb{E}\left[\left|\sum c_n X_n\right|^{4-p}\right]^{1/2}}_{\lesssim(\sum |c_n|^2)^{1/2\cdot 1/2\cdot(4-p)}}.\end{aligned}$$

So we get that

$$\left(\sum |c_n|^2\right)^{p/4} \lesssim \mathbb{E}\left[\left|\sum c_n X_n\right|^p\right]^{1/2}.$$

Now raise both sides to the power $2/p$. □

1.3 Littlewood-Paley square function estimate

Theorem 1.2 (Littlewood-Paley square function estimate)**.** *Let* $f \in \mathcal{S}(\mathbb{R}^d)$ *and define the square function*

$$S(f) = \sqrt{\sum |f_N|^2}.$$

Then

$$\|S(f)\|_p \sim_p \|f\|_p \qquad \forall 1 < p < \infty.$$

Proof. Let's prove $\|Sf\|_p \lesssim_p \|f\|_p$. Let $\{X_N\}_{n\in 2^{\mathbb{Z}}}$ be iid random variables with $X_n = \pm 1$ with equal probability. Let

$$m_X(\xi) = \sum_{N\in 2^{\mathbb{Z}}} X_N \psi_N(\xi).$$

Note that

$$m_X^{\vee} * f = \sum_{N\in 2^{\mathbb{Z}}} X_N f_N.$$

We claim that m_X is a Mikhlin multiplier uniformly in the choice of X_N.

$$|D_\xi^\alpha m_X(\xi)| \lesssim \sum_{N\in 2^{\mathbb{Z}}} N^{|\alpha|} |D_\xi^\alpha \psi|(\xi/N)$$

Since ψ has compact support on $\mathbb{R} \setminus \{0\}$, only finitely many N contribute to the sum.

$$\lesssim |\xi|^{-\alpha}. \qquad \square$$

We will finish the proof next time.

Remark 1.1. We could replace ψ by any $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and still get a Mikhlin multiplier.